THE CALCULATION OF TEMPERATURES INSIDE BUILDINGS HAVING VARIABLE EXTERNAL CONDITIONS

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Summary

Direct mathematical treatment of the thermal conditions in a building when the external temperature is not constant is generally considered too complex and recourse is made to thermal models or analogies. By the method described here, however, the mathematical treatment can be reduced to manageable proportions and, for an assumed thermal environment, the method will give a more accurate answer and will provide a more sensitive test of the effect of small changes than previous methods. An outline is given of calculations in two particular instances and the results are compared with measurements on model houses.

I. Introduction

The heat flows and the various temperatures in a building are comparatively simple to calculate when the external conditions do not vary; such calculations are treated in standard textbooks on heating and air-conditioning. However, the state of unvarying conditions is an ideal which does not occur in practice, and the simplification is valid only when the steady-state heat flow forms a large part of the total heat flow. When it does not, the effect of the distributed heat capacity of the structure is of importance and the mathematical treatment becomes very complex as reference to a text such as Carslaw and Jaeger (1947) will show. The treatment of several parallel heat paths each through possibly several slabs of material in series is exceedingly complex and is seldom if ever attempted.

The normal methods of treating the problem of house temperatures were discussed at the recent Building Research Congress: Mackey (1951) and Bruckmeyer (1951) discuss the reaction of single heat paths to particular imposed conditions but these are obviously useful only when one single element is of overwhelming importance (for example, a flat having thick heavy walls and with similar flats above and below) and Billington (1951) describes electrical analogies for use as calculating machines. Paschkis and Baker (1942) similarly describe and use an electrical analogy; Leopold (1948), Metropolitan Vickers Electrical Co. Ltd. (1950), and others describe hydraulic analogies capable of being used for heat transfer calculations. One other possible method is that of the thermal models proposed by Drysdale (1948, 1950) but this can obviously become laborious if many variables of climate and construction are to be included.

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Mathematical treatment can however be reduced to manageable proportions by a method proposed here; the method has advantages over the model and analogy methods for cases wherein the actual transfer elements do not change with time. The treatment depends on the fact that the wall of a building may be treated as a passive four-pole as is shown by van Gorcum (1950). His mathematical treatment applies equally well to the other heat paths in a building—the floor, roof, windows, and air exchange—and it will be shown here how the treatment may be extended to the combinations which occur in buildings.

II. Theory

van Gorcum shows that if an infinite homogeneous slab of material has harmonic surface temperatures T_A and T_B and heat flows at its surfaces (in the direction A to B) W_A and W_B per unit area then these are related by

$$\left.\begin{aligned} T_B &= P_{11}T_A + P_{12}W_A, \\ W_B &= P_{21}T_A + P_{22}W_A, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ (1)$$

and P_{11}, P_{12}, P_{21}, and P_{22} are related to the properties of the material by

$$\begin{aligned} P_{11} = P_{22} &= \cos \gamma l, \\ P_{12} &= -\ (\gamma l)^{-1} \sin \gamma l, \\ P_{21} &= \lambda\gamma \sin \gamma l, \end{aligned} \quad \ldots\ldots\ldots\ldots\ (2)$$

where l is the thickness of the slab, λ is the thermal conductivity of the material, c is the heat required to cause unit temperature rise of unit volume of the material, $\omega/2\pi$ is the frequency of the temperature variation, j is $(-1)^{\frac{1}{2}}$, and $\gamma^2 = -j\omega c/\lambda$.

The equations (1) can be written

$$\begin{Bmatrix} T_B \\ W_B \end{Bmatrix} = (P) \begin{Bmatrix} T_A \\ W_A \end{Bmatrix}, \quad \ldots\ldots\ldots\ldots\ (3)$$

in the notation of matrix algebra, and if the building element consists of more than one homogeneous slab in series, the matrix relating temperatures and heat flows at the outer surfaces can be found by multiplication of the matrices for each slab.

Now in a building, the interior exchanges heat with the environment via heat paths, the elements of which have thicknesses small compared with their area (except for pitched roofs, but these can also be included here since the heat capacity of the air space is low and sideways heat transfer is probably small), and hence each can be treated as a passive four-pole and an equation of the form (3) written for it. If T_A be the internal temperature, then the various external temperatures T_B will be the shade air temperature, the average sol-air temperature of the roof or the walls, or earth temperatures, depending upon the element under consideration. Further, the sum of all the heat flows to and from the interior of the building must be zero at all times and hence the equations of the type (3) may be combined by the use of the equation

$$\Sigma A\, W_A + W = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

where A is the area of the particular element, and W is the heat input from sources inside the building.

The first equation of the pair (3) may be rewritten

$$T_B/P_{12} = P_{11}/P_{12}\,.\,T_A + W_A,$$

for elements connecting with the external environment. For internal walls (matrix Q) there is no heat flow at the centre of the wall and so, taking the two sides of the wall, the second equation of (3) becomes

$$0 = Q_{21}/Q_{22}\,.\,T_A + W_A.$$

Combining these gives

$$T_A = [\Sigma A\,(T_B/P_{12}) + W]/[\Sigma A\,(P_{11}/P_{12}) + \Sigma A\,(Q_{21}/Q_{22})] \ldots \quad (5)$$

This may be compared with the equation for the steady state

$$T_A = [\Sigma A\,(T_B/R) + W]/\Sigma A\,(1/R), \quad \ldots\ldots\ldots \quad (6)$$

where the symbols have the same meaning as in (5) and R is the overall thermal resistance.

III. Calculations

The way the method works can be seen in the following examples wherein the calculated results are compared with those obtained experimentally in small models. Two models were built, both about 3 ft. square by 1 ft. high, and placed in a room the temperature of which was varied cyclically. The air temperatures of the room and of the interiors of both models were measured and the interior temperatures of the models were calculated by the method of Section II.

Both models were constructed to have various heat paths each with a different thermal conductivity and capacity. One model, illustrated in Figure 1, was constructed in masonry. It had 9.3 sq. ft. of wall of 2-in. brickwork, 1.7 sq. ft. of windows of 16-oz. glass, 7.1 sq. ft. floor area of 1-in. timber, and 7.1 sq. ft. of roof of 2-in. concrete slabs (all areas average of internal and external). An internal wall of brickwork 2 in. thick had an area of 4 sq. ft. The second (timber) model had 5.9 sq. ft. of 1-in. mineral wool with $\frac{1}{8}$-in. ply on each side, 1.7 sq. ft. of windows, and framing areas totalling 3.4 sq. ft. of $1\frac{1}{4}$-in. timber for its walls, 7.6 sq. ft. of 1-in. cane fibre board for its roof, and 7.6 sq. ft. of $\frac{1}{8}$-in. ply for its floor. Efforts were made to reduce the exchange of air between the outside and the inside of the models as far as possible since, although its effect can easily be included in the calculations, the air exchange rate could not be conveniently measured.

The values of the thermal properties (of the various materials in the models) used for calculations are set out in Table 1; the air film conductances used were 2.5 B.Th.U./sq. ft./hr./deg. F. for external surfaces

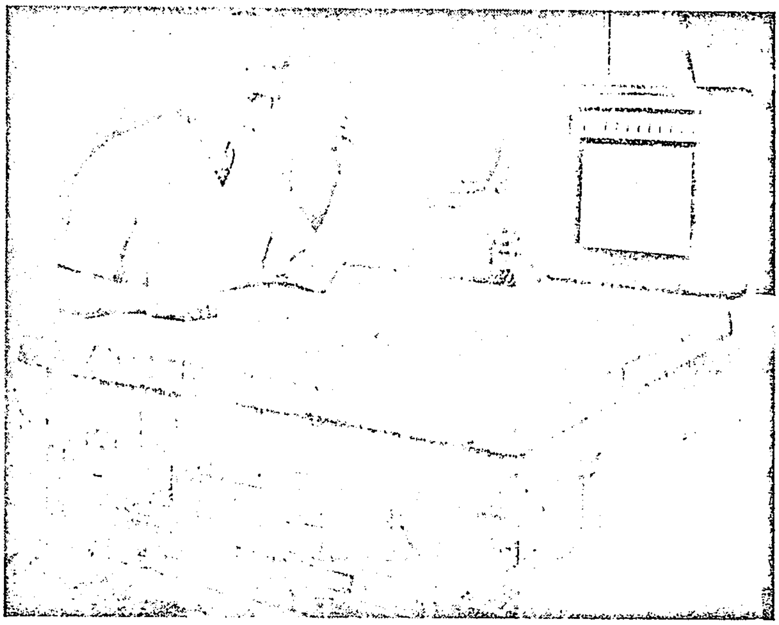

Fig. 1.—Masonry model used in measurements.

and 1.4 B.Th.U./sq. ft./hr./deg. F. for internal surfaces. Calculations of the value of A/P_{12}, AP_{11}/P_{12}, and AQ_{21}/Q_{22} gave the results of Tables 2 and 3.

TABLE 1

THERMAL PROPERTIES OF MATERIALS

Material	Conductivity (B.Th.U./sq. ft. /hr./deg. F./in.)	Specific Heat (B.Th.U./lb./deg. F.)	Density (lb./cu. ft.)
Brickwork	8	0.2	120
Concrete	10	0.2	150
Wood	1	0.3	50
Mineral wool	0.25	0.2	12
Cane fibre board	0.4	0.3	30
Window glass	High	0.2	1*

* lb./sq. ft.

The external air had an hourly temperature cycle shown graphically as the continuous line in Figure 2. Forty-eight ordinates were measured

TABLE 2
TRANSFER COEFFICIENTS FOR MASONRY MODEL

Element	Once Hourly A/P_{12}	Once Hourly AP_{11}/P_{12} or AQ_{21}/Q_{22}	Twice Hourly A/P_{12}	Twice Hourly AP_{11}/P_{12} or AQ_{21}/Q_{22}	Thrice Hourly A/P_{12}	Thrice Hourly AP_{11}/P_{12} or AQ_{21}/Q_{22}	Four Times Hourly A/P_{12}	Four Times Hourly AP_{11}/P_{12} or AQ_{22}/Q_{22}
Wall, external	$+0.6+0.6j$	$-12.1-1.0j$	$+0.3$	$-12.3-0.9j$	$+0.1-0.1j$	$-12.5-0.7j$	$-0.1j$	$-12.6-0.6j$
Wall, internal		$-12.5+0.6j$		$-12.4+0.4j$		$-12.3+0.3j$		$-12.2+0.3j$
Floor	$+0.6+0.6j$	$-7.0-1.9j$	$+0.3-0.1j$	$-7.8-1.6j$	$+0.1-0.1j$	$-8.2-1.4j$	$-0.1j$	$-8.4-1.3j$
Roof	$+0.5+0.5j$	$-9.2-0.8j$	$+0.2$	$-9.4-0.7j$	$+0.1-0.1j$	$-9.5-0.6j$	$-0.1j$	$-9.6-0.5j$
Windows	$-1.4+0.5j$	$-1.2-1.2j$	$-1.1+0.7j$	$-0.4-1.8j$	$-0.8+0.8j$	$+0.4-2.0j$	$+0.6+0.8j$	$+0.9-1.9j$
Total	$+0.3+2.2j$	$-42.0-4.3j$	$-0.3+0.6j$	$-42.3-4.6j$	$-0.5+0.5j$	$-42.1-4.4j$	$+0.6+0.5j$	$-41.9-4.0j$

TABLE 3
TRANSFER COEFFICIENTS FOR TIMBER MODEL

Element	Once Hourly A/P_{12}	Once Hourly AP_{11}/P_{12} or AQ_{21}/Q_{22}	Twice Hourly A/P_{12}	Twice Hourly AP_{11}/P_{12} or AQ_{21}/Q_{22}	Thrice Hourly A/P_{12}	Thrice Hourly AP_{11}/P_{12} or AQ_{21}/Q_{22}	Four Times Hourly A/P_{12}	Four Times Hourly AP_{11}/P_{12} or AQ_{21}/Q_{22}
Wall, external timber	$+0.2+0.1j$	$-3.4-0.9j$	$-0.1j$	$-3.8-0.7j$		$-3.9-0.6j$		$-4.0-0.5j$
Wall, external insulated	$+0.3+0.5j$	$-4.4-3.0j$	$+0.2$	$-6.2-2.7j$	$-0.1j$	$-6.8-2.1j$		$-7.0-1.5j$
Floor	$-5.8+1.6j$	$-6.6-1.0j$	$-4.9+2.7j$	$-7.1-1.7j$	$-3.7+3.3j$	$-7.6-2.4j$	$-2.8+3.3j$	$-8.3-2.0j$
Windows	$-1.4+0.5j$	$-1.2-1.2j$	$-1.1+0.7j$	$-0.4-1.8j$	$-0.8+0.8j$	$+0.4-2.0j$	$-0.6+0.8j$	$+0.9-1.9j$
Roof	$+0.5+0.5j$	$-5.3-2.3j$	$+0.1-0.1j$	$-6.4-2.2j$	$-0.1j$	$-7.5-1.8j$		$-8.9-0.6j$
Total	$-6.2+3.2j$	$-20.9-8.4j$	$-5.7+3.3j$	$-23.9-9.1j$	$-4.5+3.9j$	$-25.4-8.9j$	$-3.4+4.1j$	$-27.3-6.5j$

and a Fourier analysis made for the first five harmonics. The external cycle obtained was

$$T_B = 23.9 + 20.0 \sin(2\pi x + 1.21) + 2.2 \sin(4\pi x + 1.24) + 3.5 \sin(6\pi x + 1.90) + 1.4 \sin(8\pi x + 2.14), \quad \ldots\ldots\ (7)$$

where x is the time in hours and T_B is given in recorder divisions. Synthesis from these harmonics gave the broken curve of Figure 2.

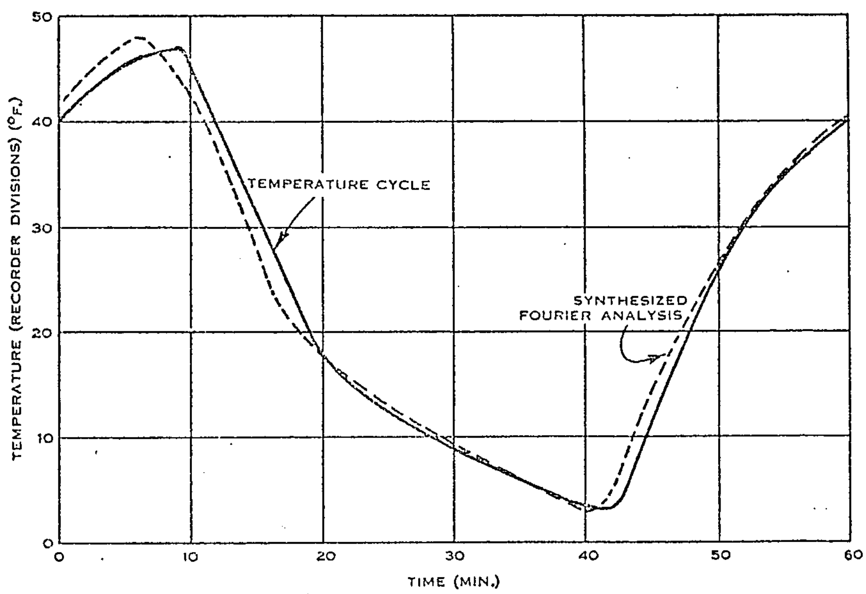

Fig. 2.—Comparison of actual external temperature and that used for calculations.

For each frequency the values from Tables 2 and 3 and of T_B were substituted in equation (5) and the calculated internal temperatures were

Masonry model:

$$T_A = 23.9 + 1.1 \sin(2\pi x - 0.60) + 0.1 \sin(6\pi x - 0.76). \quad \ldots\ (8)$$

Timber model:

$$T_A = 23.9 + 6.1 \sin(2\pi x + 0.39) + 0.6 \sin(4\pi x - 0.53) + 0.8 \sin(6\pi x - 1.25) + 0.3 \sin(8\pi x - 2.27) +. \quad \ldots\ldots\ (9)$$

These are plotted in Figure 3 (the broken lines) and are tabulated in Table 4. The results obtained experimentally are shown as continuous lines in the same figure and are given for comparison in Table 4. These experimental and calculated results can be seen to be in good agreement. The differences that do occur are probably less than might be expected when it is realized that the thermal properties of the materials could well

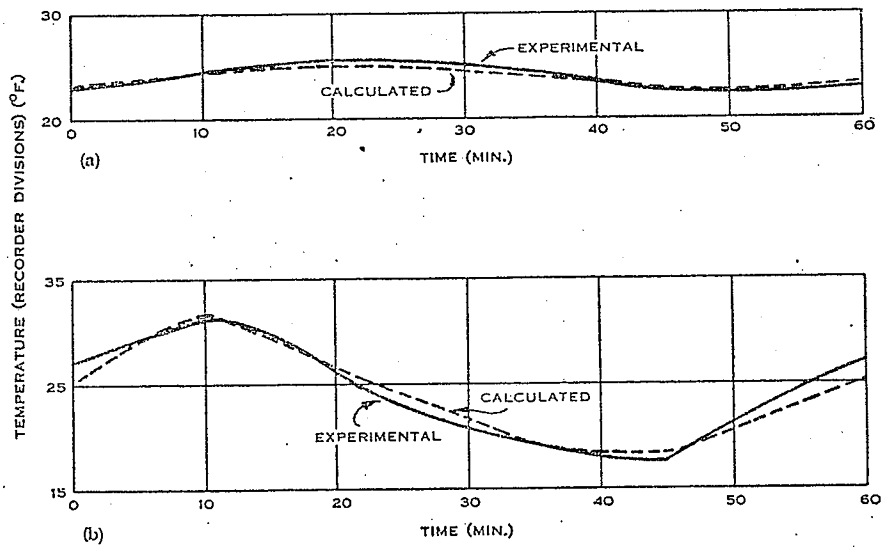

Fig. 3.—Comparison of results.
(*a*) Masonry model.
(*b*) Wooden model.

be 10 per cent. in error and the film coefficients possibly more, and that recorder errors of the order of one-half scale division could be expected.

TABLE 4

COMPARISON OF TEMPERATURES

All temperatures given in recorder divisions

Time (min.)	Temperature					
	External		Masonry Model		Timber Model	
	Experimental	Fourier Synthesis	Experimental	Calculated	Experimental	Calculated
5	46.1	47.1	23.8	23.9	29.0	29.2
10	43.6	43.1	24.6	24.4	31.1	31.4
15	26.9	26.2	25.1	24.7	29.6	29.4
20	17.8	17.8	25.3	24.9	26.0	26.5
25	12.8	13.1	25.3	24.9	23.0	24.7
30	9.2	9.5	25.0	24.5	21.0	21.8
35	6.4	6.4	24.6	24.0	19.4	19.0
40	4.2	4.0	23.6	23.4	18.1	18.8
45	15.6	16.1	22.9	23.1	17.6	18.6
50	29.1	29.3	22.7	22.9	21.1	20.6
55	36.2	36.3	22.8	22.9	24.3	22.6
60	40.7	41.0	23.0	23.3	27.0	25.0

IV. CONCLUSIONS

Heat transfer networks of several paths unvarying with time and having periodic variations of temperature can be studied by direct mathematical calculation; each elemental path is considered as a passive four-pole net and laws similar to Kirchhoff's network laws of electricity used to combine their effects. Although somewhat lengthy calculations are needed in complex cases, the transfer coefficients for a particular building element do not need to be calculated each time, and overall, the work will be far less than for model or analogy methods. The effect on internal temperatures of small changes in the network can be determined more simply than by other methods and the results will be far more accurate.

V. References

Billington, N. S. (1951).—The use of electrical analogies in heating research. Build. Res. Congr. 1951 Pap. Div. 3. pp. 75-81.

Bruckmeyer, F. (1951).—Cooling and warming of buildings. Build. Res. Congr. 1951 Pap. Div. 3. pp. 66-74.

Carslaw, H. S., and Jaeger, J. C. (1947).—"Conduction of Heat in Solids." (Oxford Univ. Press.)

Drysdale, J. W. (1948).—Thermal characteristics of model structures. Australia. Commonw. Exp. Build. Sta. Tech. Study 26.

Drysdale, J. W. (1950).—The thermal behaviour of buildings. Australia. Commonw. Exp. Build. Sta. Tech. Study 33.

van Gorcum, A. H. (1950).—Theoretical considerations on the conduction of fluctuating heat flow. *Appl. Sci. Res. Hague* A 2: 272-80.

Leopold, C. S. (1948).—Hydraulic analogue for the solution of problems of thermal storage, radiation conduction, and convection. *Heat. Pip. Air Condit.* 20: 105-11.

Mackey, C. O. (1951).—The steady periodic state—a review of some computational methods. Build. Res. Congr. 1951 Pap. Div. 3. pp. 60-5.

Metropolitan Vickers Electrical Co. Ltd. (1950).—Analogy model for the solution of unsteady-state heat conduction problems. "Handbook of Scientific Instruments and Apparatus 1950." pp. 239-40. (Physical Society: London.)

Paschkis, V., and Baker, H. D. (1942).—A method for determining unsteady-state heat transfer by means of an electrical analogy. *Trans. Amer. Soc. Mech. Engrs.* 64: 105-12.